AN ITERATIVE SOLUTION OF THE
ROUGH-SURFACE SCATTERING PROBLEM

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Final Report

Contract N00014-95-1-0538

February 4, 1997

DTIC QUALITY INSPECTED 4

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Abstract

An iterative solution to the problem of scattering from a one-dimensional rough surface is obtained for the Dirichlet boundary condition. The advantages of this method are that bounds for the convergence of the solution may be established and that the solution may be readily iterated to sufficiently high order in the interaction to examine the rate at which it converges. Absolute convergence of the iterative solution is also a sufficient condition for the convergence of the operator expansion method for surfaces on which the slope is everywhere less than unity. A numerical example of scattering from an echelette grating is considered, and bounds for convergence established. It is found that for scattering from such surfaces, the rate at which the iterative solution converges decreases as the surface slope is increased. Corresponding results are found for the operator expansion method.

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1. Introduction

In the past two decades, many new approaches to the rough-surface scattering problem have been proposed that have potential for treating scatter from very rough surfaces. These include the small-slope approximation (Voronovich 1985) and its extensions (Tatarskii 1993, McDaniel 1995, and Charnotskii and Tatarskii 1996). Also noteworthy in this respect are the phase perturbation technique of Shen amd Maradudin (1980) and the development of second order corrections to the Kirchhoff approximation by Chen and Ishimaru (1990). The operator expansion method of Milder (1991) may also be applicable to very rough surfaces. This method is basically an extension of earlier work by Lysanov (1956) and Meecham (1956) to obtain higher order terms of a series solution to the surface scattering problem (Milder 1996). In a further extension, Voronovich (1996) has exploited the fact that the operator expansion method is not gauge invariant to formulate the non-local small-slope approximation.

It is not surprising that operator expansion methods are more accurate than small-slope approximations since the small-slope approximations are obtained through the expansion of certain terms in the operator expansions (McDaniel 1995). For scatter from random-rough surfaces, the small-slope approximations permit the calculation of ensemble averages, while the operator expansion methods require Monte Carlo simulation. However, such simulations may be readily performed through the repeated application of the Fourier transform (Kaczkowski and Thorsos 1994). An important remaining issue is the regime of applicability of operator expansions.

In this article, an iterative solution to an integral equation of the first kind for the normal derivative of the surface field is obtained for scattering from a one-dimensional randomly rough surface characterized by the Dirichlet boundary condition. This results in a series for the surface potential for which the conditions for convergence may readily be established. Successive terms in this series may also be easily generated to examine the rate at which the series converges. A disadvantage of this solution is that obtaining numerical results

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requires even more computational effort than in the case of the operator expansion method. In addition to the need for Monte Carlo simulation for treating randomly rough surfaces, double integrations are required for scatter from one-dimensional surfaces, and four-fold integrations for the two-dimensional case. Because operator expansions are obtained by series expansion of terms in the iterative solution, a sufficient condition for the convergence of the operator expansion method may be obtained for surfaces having slopes everywhere less than unity where this series expansion is valid.

In section 2 of this paper, the basic theoretical development of the iterative solution is presented and its relationship to the corresponding operator expansion is discussed. The surface potential for a plane wave incident on a flat surface of arbitrary slope is evaluated to demonstrate the differences between the iterative solution and operator expansion. Section 3 addresses the numerical solution of the relevant equations and establishes the conditions for convergence of the iterative and operator series.

Scattering from a echelette grating is considered in section 4, in which numerical results for the iterative solution and operator expansion are compared with exact solutions. First, the region of convergence of the iterative solution is established and is found to significantly exceed that of Milder's (1991) Fresnel phase criterion. The rate of convergence of the iterative solution is examined by computing successive terms of this series and is found to depend on the incident grazing angle as well as the surface slope. Numerical results are also presented for the operator expansion method. In section 5, the findings of this study are discussed.

2. Theoretical development

The problem addressed is illustrated in figure 1 in which a plane wave p_{inc} of wavelength $\lambda = 2\pi/k$

$$p_{inc} = \exp[i(\kappa_0 x - \nu_0 z)] \tag{1}$$

is incident on a one-dimensional rough surface on which the Dirichlet boundary condition is obeyed. Here $\kappa_0 = k \cos \theta_0$, $\nu_0 = k \sin \theta_0$, θ_0 is the incident grazing angle, and k is the wave number.

The scattered field p_s is represented as

$$p_s = \int d\kappa_f S(\kappa_f, \kappa_0) \exp[i(\kappa_f x + \nu_f z)] \tag{2}$$

where $\kappa_f = k\cos\theta_f$, $\nu_f = k\sin\theta_f$, and the scattering amplitude $S(\kappa_f, \kappa_0)$ is defined as

$$S(\kappa_f, \kappa_0) = k \int \frac{dx}{4\pi\nu_f} \exp\{-i[\kappa_f x + \nu_f \zeta(x)]\}\psi(x) \tag{3}$$

where the surface potential $\psi(x)$ is defined below. The definition of $S(\kappa_f, \kappa_0)$ in (3) differs from that of Voronovich (1985), and the requirement that it be reciprocal takes the form $\nu_f S(\kappa_f, \kappa_0) = \nu_0 S(-\kappa_0, -\kappa_f)$.

Evaluation of the scattering amplitude requires the solution of an integral equation (Uretsky 1965) for the surface potential

$$\int \psi(x') V(x, x') dx' = -2\widehat{\psi}(x) \tag{4}$$

where

$$\psi(x) = \frac{i}{k} \left[-\frac{\partial p}{\partial z} + \zeta'(x)\frac{\partial p}{\partial x} \right]_{z=\zeta(x)} \tag{5}$$

$$V(x, x') = kH_0^{(1)}(k\rho)/2 \tag{6}$$

with $\rho = \{(x-x')^2 + [\zeta(x) - \zeta(x')]^2\}^{1/2}$, $H_0^{(1)}$ a Hankel function of the first kind, ζ' the surface slope, and $\widehat{\psi}(x) = p_{inc}(x)$. In (4), $\widehat{\psi}(x)$ is evaluated on the rough surface $\zeta(x)$. Applying the operator $\int dx \exp(-i\kappa x)$ to both sides of (4) yields

$$\int \frac{dx\,dx'}{2\pi} V(x, x') \exp(-i\kappa x) \int d\kappa' \psi(\kappa') \exp(i\kappa' x') = -2\widehat{\psi}(\kappa) \tag{7}$$

where $\psi(\kappa) = \int dx \psi(x) \exp(-i\kappa x)$, and a similar expression holds for $\widehat{\psi}(\kappa)$.

To proceed, we write

$$V(x, x') = V^{(0)}(x, x') - V^{(1)}(x, x') \tag{8}$$

where $V^{(0)}(x, x') = kH_0^{(1)}(k|x - x'|)/2$, and $V^{(1)}(x, x') = V^{(0)}(x, x') - V(x, x')$. The surface potential $\psi(\kappa)$ is next expanded as an ordered series in the interaction

$$\psi(\kappa) = \sum_n \psi^{(n)}(\kappa) = \psi^{(1)}(\kappa) + \psi^{(2)}(\kappa) + \psi^{(3)}(\kappa) + \cdots. \tag{9}$$

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Substituting (8) and (9) in (7), and solving successively for the $\psi^{(i)}(\kappa)$ yields

$$\int \psi^{(1)}(\kappa')A(\kappa,\kappa')d\kappa' = -2\widehat{\psi}(\kappa) \tag{10}$$

where

$$A(\kappa,\kappa') = \int \frac{dx\,dx'}{2\pi}V^{(0)}(x,x')\exp[i(\kappa'x' - \kappa x)]. \tag{11}$$

The integral on the left hand side of (10) may be evaluated to obtain

$$\psi^{(1)}(\kappa) = -2\nu\widehat{\psi}(\kappa)/k \tag{12}$$

where

$$\nu = \nu(\kappa) = \left\{ \begin{array}{ll} (k^2 - \kappa^2)^{1/2}; & \kappa \leq k \\ i(\kappa^2 - k^2)^{1/2}; & \kappa > k. \end{array} \right.$$

Higher order terms for which $n > 1$, take the form

$$k\psi^{(n)}(\kappa)/\nu = \int \psi^{(n)}(\kappa')A(\kappa,\kappa')d\kappa' = \int \psi^{(n-1)}(\kappa')B(\kappa,\kappa')d\kappa' \tag{13}$$

where $B(\kappa,\kappa')$ is defined similarly to $A(\kappa,\kappa')$ in terms of $V^{(1)}(x,x')$. Uretsky (1965) has shown that multiple integrals of the type appearing on the left hand side of (7) converge provided $\psi(\kappa')$ grows no faster than some finite power of κ', so that the interchange of the integral over κ' with the spatial integrals to obtain (10) and (13) is permissable.

To obtain the operator expansion, $H_0^{(1)}(k\rho) - H_0^{(1)}(k|x - x'|)$ is expanded using (Watson 1966)

$$H_0^{(1)}|z(1 + r)^{1/2}| - H_0^{(1)}|z| = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!}\left(\frac{r|z|}{2}\right)^m H_m^{(1)}|z|. \tag{14}$$

The convergence of this series is governed by the singular nature of the Hankel function. Inserting the limiting form of the Hankel function into (14), it follows that the series is absolutely and uniformly convergent provided $r = [\zeta(x) - \zeta(x')]^2/(x - x')^2 < 1$, or that, the surface slope is everywhere less than unity. Substituting (14) into (9) yields an expansion for the surface potential. If (9) and (14) are absolutely convergent, the summation of this expansion may be rearranged in any arbitrary order since the orders of integration and summation in (9) may be freely interchanged. In particular, the sum may be rearranged

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to form a simple series in powers of the surface excursion to obtain an operator expansion. Thus, under these conditions, the operator expansion will converge absolutely.

The operator expansion may also be obtained by substituting (14) directly into (7). Expanding the surface potential $\psi(\kappa)$ in an ordered series in the surface elevation

$$\psi(\kappa) = \psi_1(\kappa) + \psi_2(\kappa) + \psi_3(\kappa) + \cdots \tag{15}$$

and also inserting it into (7), yields an expression that may be solved successively to determine the ψ_i. In this case, the spatial integrals can be performed to obtain (McDaniel 1995)

$$\psi_1(\kappa) = -2\nu\widehat{\psi}(\kappa)/k \tag{16}$$

and

$$2\psi_2(\kappa) = \nu \int d\kappa' \psi_1(\kappa')\delta(\kappa' + q + q' - \kappa) \int dq\,dq'\tilde{\zeta}(q)\tilde{\zeta}(q')F(q,q') \tag{17}$$

where

$$F(q,q') = \nu(\kappa' + q + q') + \nu(\kappa') - \nu(\kappa' + q) - \nu(\kappa' + q')$$

and $\tilde{\zeta}(q)$ is the Fourier transform of the surface displacement $\zeta(x)$

$$\tilde{\zeta}(q) = \int \frac{dx}{2\pi}\zeta(x)\exp(-iqx).$$

On comparing (12) and (16), it is evident that the leading terms of the iterative solution and the operator expansion are identical.

Because $F(0,q') = F(q,0) = 0$ in (17), it may be argued (Voronovich 1985) that $\psi_2(\kappa)$ is of second and higher order in the surface slope. Thus, following Voronovich (1985, 1996), an integration by parts may be performed to extract the contributions of second order in the slope from (17)

$$2\tilde{\psi}_2(\kappa) = ik \int d\kappa' \psi_1(\kappa')\tilde{\zeta}(\kappa - \kappa')[\nu + \nu_0 - \nu(\kappa') - \nu(\kappa - \kappa')]/\nu(\kappa'). \tag{18}$$

This result will be used in examining the convergence of the operator expansion method.

The differences between the iterative and operator series are apparent even in very simple cases such as scattering from a flat surface of slope s for which $\zeta(x) = sx$. The exact solution $\psi_E(x)$ for the surface potential is

$$\psi_E(x) = -2\exp(i\tilde{\kappa}x)(\tilde{k}^2 - \tilde{\kappa}^2)^{1/2}/k \tag{19}$$

where $\tilde{k}^2 = k^2(1 + s^2)$ and $\tilde{\kappa} = \kappa_0 - \nu_0 s$. The iterative solution yields

$$\psi^{(1)}(x) + \psi^{(2)}(x) = -2\exp(i\tilde{\kappa}x)(k^2 - \tilde{\kappa}^2)^{1/2}[2 - (k^2 - \tilde{\kappa}^2)^{1/2}/(\tilde{k}^2 - \tilde{\kappa}^2)^{1/2}]/k. \tag{20}$$

Finally, the operator expansion yields

$$\psi_1(x) + \psi_2(x) = -\exp(i\tilde{\kappa}x)(k^2 - \tilde{\kappa}^2)^{1/2}[2 + s^2k^2/(k^2 - \tilde{\kappa}^2)]/k. \tag{21}$$

The surface potential for the iterative solution (20) may be obtained from (19) by writing $(\tilde{k}^2 - \tilde{\kappa}^2)^{1/2} = (k^2 - \tilde{\kappa}^2)^{1/2} + (\tilde{k}^2 - \tilde{\kappa}^2)^{1/2}[1 - (k^2 - \tilde{\kappa}^2)^{1/2}/(\tilde{k}^1 - \tilde{\kappa}^2)^{1/2}]$ and expanding the coefficient of the second term on the right hand side of this expression for small s. Further expansion of (20) for small s yields (21). Because the next higher order terms in these expansions are of fourth order in the slope, both (20) and (21) are valid to third order in the slope.

The behavior of (20) and (21) as a function of the incident grazing angle θ_0 differs significantly: (21) is unbounded when θ_0 or θ_f vanishes irrespective of the slope, while (20) vanishes at these points. On the other hand, (20) is unbounded for glancing incidence on the surface where $\tan\theta_0 = -s$ and $\psi_E = 0$. Because higher order terms in the iterative solution contain successively higher order powers of $(\tilde{k}^2 - \tilde{\kappa}^2)^{-1/2}$, it is clear that the iterative solution cannot be truncated at arbitrary order, for this example where the surface excursion is unbounded. It is thus anticipated that, not only the slope, but the rough surface excursion as well, will play a role in the series convergence.

3. Numerical solution

This section addresses the numerical evaluation of the iterative solution and the conditions for its convergence. The random-rough surface $\zeta(x)$ is treated as periodic with a

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fundamental period Λ so that scatter occurs at grating angles θ_n where

$$\cos\theta_n = \cos\theta_0 + 2\pi n/(k\Lambda). \tag{22}$$

It is convenient also to define $\kappa_n = k\cos\theta_n$ and $\nu_n = \nu(\kappa_n)$.

The integral equation for the surface potential (7) is replaced by the matrix expression (Holford 1981)

$$\mathbf{M}\psi = -2\widehat{\psi} \tag{23}$$

where

$$\widehat{\psi}_n = \int_0^\Lambda \frac{dx}{\Lambda} p_{inc} \exp(-i\kappa_n x)$$

$$\psi_n = \int_0^\Lambda \frac{dx}{\Lambda} \psi(x) \exp(-i\kappa_n x)$$

$$M_{mn} = \int_0^\Lambda \frac{dx}{\Lambda} \int_{-\infty}^\infty dx' V(x,x') \exp[i(\kappa_n x' - \kappa_m x)].$$

Using (8) to define $\mathbf{A}$ and $\mathbf{B}$ yields

$$A_{mn} = k\delta_{m-n}/\nu_m \tag{24}$$

$$\mathbf{B} = \mathbf{A} - \mathbf{M}. \tag{25}$$

With (24) and (25), (23) may be rewritten as

$$\psi - \mathbf{A}^{-1}\mathbf{B}\psi = -2\mathbf{A}^{-1}\widehat{\psi}. \tag{26}$$

The exact solution to (26) is

$$\psi_E = -2(1 - \mathbf{C})^{-1}\mathbf{A}^{-1}\widehat{\psi} \tag{27}$$

and the iterative solution is

$$\psi = -2(1 + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots)\mathbf{A}^{-1}\widehat{\psi} \tag{28}$$

where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$.

The conditions under which ψ_E converges as the dimensions of the matrix $\mathbf{C}$ are increased have been investigated by Urusovskii (1965) and McDaniel and Krauss (1991). The concern here is the condition under which the series $1 + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots$ converges to $(1 - \mathbf{C})^{-1}$. The requirement for convergence is that the spectral radius of $\mathbf{C}$ be less than unity (Varga 1962), where the spectral radius $\rho(\mathbf{C}) = |\lambda_1|$ and λ_1 is the eigenvalue of the $n \times n$ complex matrix $\mathbf{C}$ of greatest absolute value. In this case, the series $\psi_n = \sum_j \psi_n^{(j)}$ converges absolutely. Numerical studies are necessary to further pursue conditions for the convergence of the iterative and operator series.

4. A numerical example

Numerical results are obtained for scattering from a periodic grating described by

$$\zeta(x) = \begin{cases} \pi a(1 - 4x/\Lambda)/2; & 0 \leq x \leq \Lambda/2 \quad \text{facet 1} \\ \pi a(-3 + 4x/\Lambda)/2; & \Lambda/2 \leq x \leq \Lambda \quad \text{facet 2.} \end{cases} \tag{29}$$

The slope of the grating's facets $\tan\epsilon = 2\pi a/\Lambda$, and its rms roughness $\sigma = \pi a/\sqrt{12}$.

First, let us consider the range of the parameters $k\sigma$ and $\tan\epsilon$ for which the iterative solution converges. For this surface, the elements of the matrix $\mathbf{M}$ derived in the appendix are given by

$$M_{mn} = (2ka/\pi^2) \sum_j \frac{a\nu_{n+j}[1 - (-1)^j \exp(i\nu_{n+j}a\pi)]}{[j^2 - a^2\nu_{n+j}^2][(n - m + j)^2 - a^2\nu_{n+j}^2]}$$
$$- (ika/\pi) \sum_j (\delta_{n-m} + \delta_{n-m+2j})/(j^2 - a^2\nu_{n+j}^2). \tag{30}$$

Figure 2 shows values of the two parameters for which $\rho(\mathbf{C})$ is unity for various angles of incidence. For $\tan\epsilon \leq 1.5$, all of the points lie to the right of the line $k\sigma = \pi/\tan\epsilon$ suggesting that this line is an upper bound on the parameters that will always yield convergent iterative solutions for low to moderate surface slopes.

Figure 3 shows values of the spectral radius computed along the line $k\sigma = \pi/\tan\epsilon$ for two values of the incident grazing angle $\theta_0 = \pi/2,\ \pi/4$. Here, again it appears that for surfaces of low to moderate slope a sufficient condition for convergence of the iterative solution is that $k\sigma \tan\epsilon < \pi$.

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For slopes of less than unity, the operator expansion method will also converge in this region by virtue of the arguments presented in sections 2 and 3. This bound may be applicable to scattering from other surfaces as well. Kaczkowski (1994) applied the operator expansion method to random-rough surfaces characterized by a Gaussian wavenumber spectrum. For a surface having an rms slope $\sigma' = 0.85$, and an incident grazing angle of $60°$ he found for $k\sigma\sigma' = 3.56$ that the method converged for $40° \geq \theta_f \geq 140°$ and appeared to diverge for other scattering angles. Kaczkowski also reported that for higher values of $k\sigma\sigma'$, the operator expansion method failed to converge.

If the iterative solution or the operator expansion are to be useful in practical applications, they must not only converge, but converge rapidly. To examine the convergence of these series, an example presented by Voronovich (1996) is considered for which $\tan \epsilon = 0.318$ and $\theta_0 = 0.609$. In this case, $k\sigma$ has been reduced to 9.4 to insure convergence. The reflection coefficients shown in figure 4 obey

$$R_n = 2\pi S(\kappa_n, \kappa_0) = (k/2\nu_n) \sum_m \psi_m \int_0^\Lambda \frac{dx}{\Lambda} \exp[ix(\kappa_m - \kappa_n) - i\nu_n \zeta(x)]. \tag{31}$$

In figure 4, predictions of the iterative solution are compared to the exact solution – with the numbers in parentheses denoting the order in the slope to which the predictions are valid. It is evident from this comparison, that terms of 5th order in the slope must be retained to obtain good agreement with the exact solution.

Corresponding results are shown for the operator expansion in figure 5, in which the numbers in parentheses again indicate the order in the slope of the predictions. For the periodic surface considered here (17) and (18) take the form

$$2\psi_{2_m} = \nu_m \sum_{j,n} \psi_{1_j} \tilde{\zeta}_n \tilde{\zeta}_{m-n-j} (\nu_m + \nu_j - \nu_{j+n} - \nu_{m-n}) \tag{32}$$

$$2\tilde{\psi}_{2_m} = i\nu_m \sum_j \psi_{1_j} \tilde{\zeta}_{m-j} (\nu_m + \nu_0 - \nu_j - \nu_{m-j})/\nu_j \tag{33}$$

where

$$\tilde{\zeta}_n = \int_0^\Lambda \frac{dx}{\Lambda} \zeta(x) \exp(-2\pi inx/\Lambda) = 2a/(\pi n^2); \quad n \text{ odd.}$$

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The operator expansion appears to converge more rapidly than the iterative solution near the forward scattering peak at $n = 0$ which corresponds to double scattering, first by facet one and thence by facet two. The peak at $n = -15$ is due to single scatter from facet two. Neither series converges rapidly when the scattered field is near grazing where $n > 0$.

The predictions of the iterative solution for scattering from a surface on which $\tan \epsilon$ is greater than unity are shown in figure 6. In this example, $k\sigma = 2$ and $\tan \epsilon = 1.2$. The magnitude of the reflection coefficient R_{-1} converges rapidly when the incident field is near grazing, $\theta_0 < 30°$, but at higher angles, terms of ninth order in the slope must be retained for agreement with the exact solution. For completeness, the corresponding predictions of the operator expansion method are shown in figure 7. In this case there is no gaurantee that results are convergent, however, retaining terms of second and third order in the slope appears to provide better agreement with the exact solution – especially at higher grazing angles.

5. Discussion

This paper has addressed the two-dimensional problem of scattering from a rough surface on which the Dirichlet boundary condition is imposed. The method of successive approximations has been applied to an integral equation of the first kind to obtain an iterative solution for the surface potential. The conditions under which this iterative solution converges have been established, and it has been shown that this regime of convergence is shared also by the operator expansion method with the only additional provision being that the slope of the rough surface is always less than unity. While the conditions for convergence obtained in a numerical example are generous compared to Milder's (1991) Fresnel phase criterion $k\sigma\sigma' \ll 1$, where σ' is the rms surface slope, they are still quite restrictive. Milder's criterion may, however, play a role in the rate at which operator expansions converge.

McDaniel (1995) has shown that small-slope expansions may be obtained by expanding terms of the type $\exp\{i\nu_{0,f}[\zeta(x) - \zeta(x')]\}$ of the operator expansion's *scattering amplitude*. The arguments of section 2 are not applicable in this case, and further studies are necessary to

determine the range of parameters for which this series converges. The last approximation that we consider in this hierarchy of solutions is perturbation theory, which results from expanding the term $\exp[i(\nu_f + \nu_0)\zeta(x)]$ of the *scattering amplitude* obtained from of the small-slope approximation. The arguments of section 2 are again inapplicable, however, it is clear in this instance, that the rate of convergence of the solution will be strongly dependent on the magnitude of the exponential's argument.

Currently, the region of convergence of these last two solutions and the rate at which they converge can only be determined through numerical studies, which are severely limited by the complexity of higher order terms in these approximations. For example, the perturbation based small-slope approximation requires the evaluation of 27 terms to obtain a result valid to third order in the slope (Thorsos and Broschat 1995) and McDaniel's (1995) extended small-slope approximation contains more than 100 terms at this order in the slope. Because higher order terms in the iterative solution may be readily generated, numerical studies based on this series may provide insight into the convergence rates of the other approximations. As an example, the slow convergence in figure 4 for $n > 0$, may lead to slow convergence of the operator and small-slope series when the scattered field is near grazing.

Acknowledgement

Appendix

The elements of the matrix **M** are evaluated as in Uretsky's (1965) treatment of scattering by a sinusoidal surface. Here, the Hankel function appearing in the matrix will be represented by

$$H_0^{(1)}(k\rho) = \int_{-\infty}^{\infty} \frac{dk_x}{\pi \nu_x} \exp[ik_x(x - x') + i\nu_x|\zeta(x) - \zeta(x')|] \qquad (A1)$$

where $\nu_x = \nu(k_x)$. Noting that $\exp[i\nu_x|\zeta(x) - \zeta(x')|]$ is a periodic function of x' for fixed x it may be expressed as

$$\exp[i\nu_x|\zeta(x) - \zeta(x')|] = \sum_j C_j(x, \nu_x) \exp(2\pi i j x'/\Lambda) \qquad (A2)$$

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where

$$C_j(x, \nu_x) = \int_0^\Lambda \frac{dx''}{\Lambda} \exp[i\nu_x|\zeta(x) - \zeta(x'')| - 2\pi ijx''/\Lambda]. \tag{A3}$$

The matrix elements M_{mn} then take the form

$$M_{mn} = (k/2\pi) \int_0^\Lambda \frac{dx}{\Lambda} \exp(-i\kappa_m x) \int_{-\infty}^\infty dx' \exp(i\kappa_n x')$$

$$\times \int_{-\infty}^\infty \frac{dk_x}{\nu_x} \exp[ik_x(x - x')] \sum_j C_j(x, \nu_x) \exp(2\pi ijx'/\Lambda). \tag{A4}$$

Performing the integral over x' in this expression yields

$$M_{mn} = k \sum_j \int_0^\Lambda \frac{dx}{\Lambda} \exp[2\pi ix(n + j - m)/\Lambda]C_j(x, \nu_{n+j})/\nu_{n+j} \tag{A5}$$

where the interchange of the summation and integration to obtain (A5) has been justified by Uretsky (1965). Using (A3) in (A5), the spatial integrals may readily be performed to obtain

$$M_{mn} = M_{nm}^{(1)} + M_{nm}^{(2)}$$

where

$$M_{mn}^{(1)} = (2ka/\pi^2) \sum_j \frac{a\nu_{n+j}[1 - (-1)^j \exp(i\nu_{n+j}a\pi)]}{[j^2 - a^2\nu_{n+j}^2][(n - m + j)^2 - a^2\nu_{n+j}^2]} \tag{A6}$$

$$M_{mn}^{(2)} = -(ika/\pi) \sum_j \frac{\delta_{n-m} + \delta_{n-m+2j}}{j^2 - a^2\nu_{n+j}^2}. \tag{A7}$$

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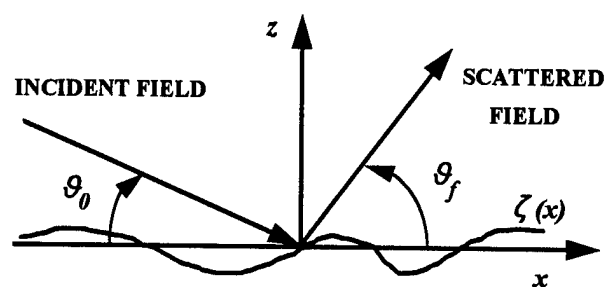Watson G N 1966 *Theory of Bessel Functions* Cambridge p 141

Figure 1. Geometry for scattering from a random-rough one-dimensional surface.
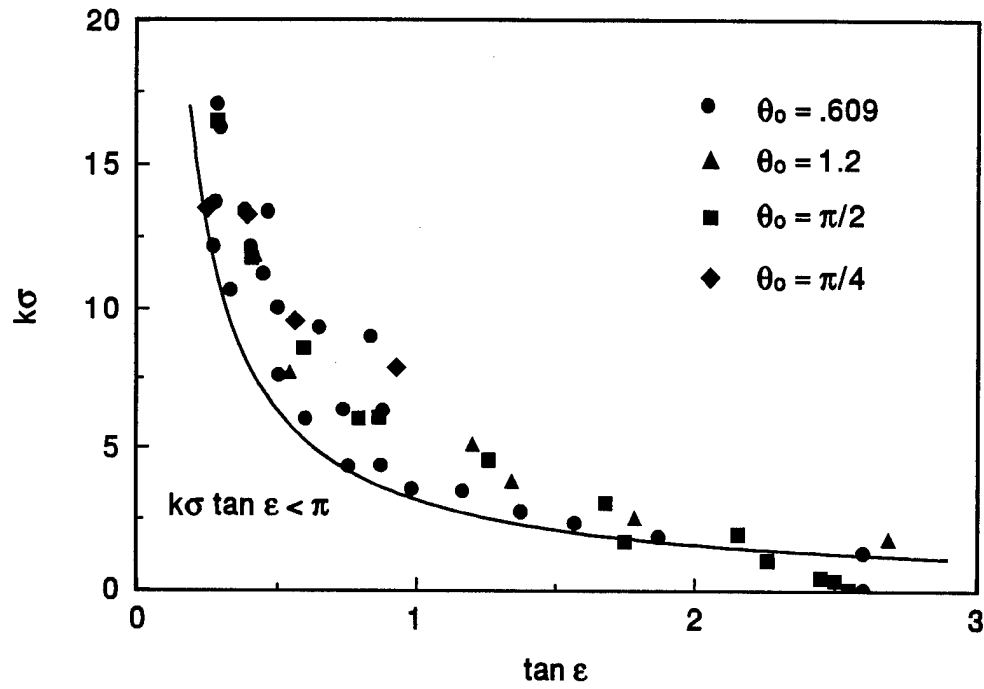
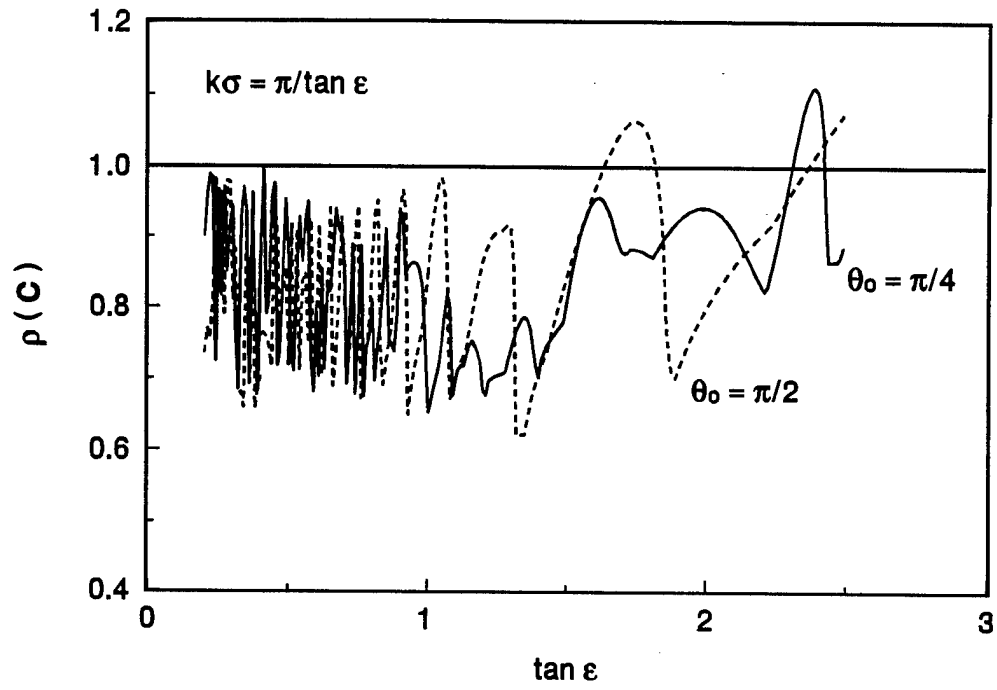**Figure 2.** Values of $k\sigma$ and $\tan\epsilon$ for which the spectral radius $\rho(\mathbf{C})$ is unity.

Figure 3. Spectral radius $\rho(\mathbf{C})$ as a function of $\tan \epsilon$ for $k\sigma = \pi / \tan \epsilon$.
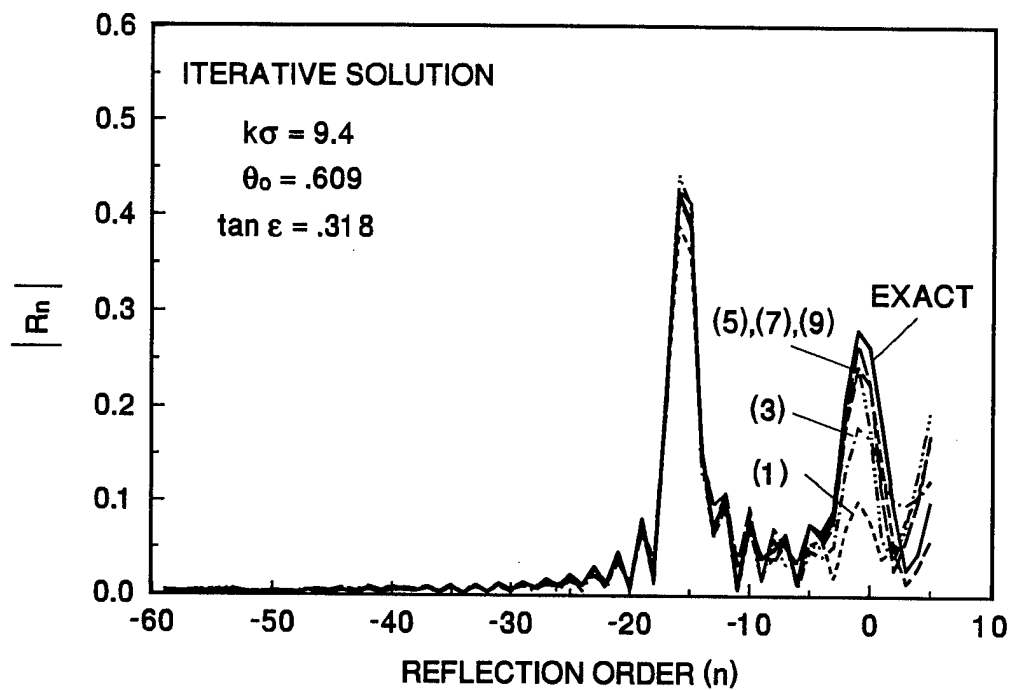
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Figure 4. Magnitude of the reflection coefficient as a function of reflected order predicted by the iterative solution for $k\sigma = 9.4$, $\tan \epsilon = .318$, and an incident grazing angle $\theta_0 = .609$. The numbers in parentheses indicate the order of the slope to which the predictions are valid.
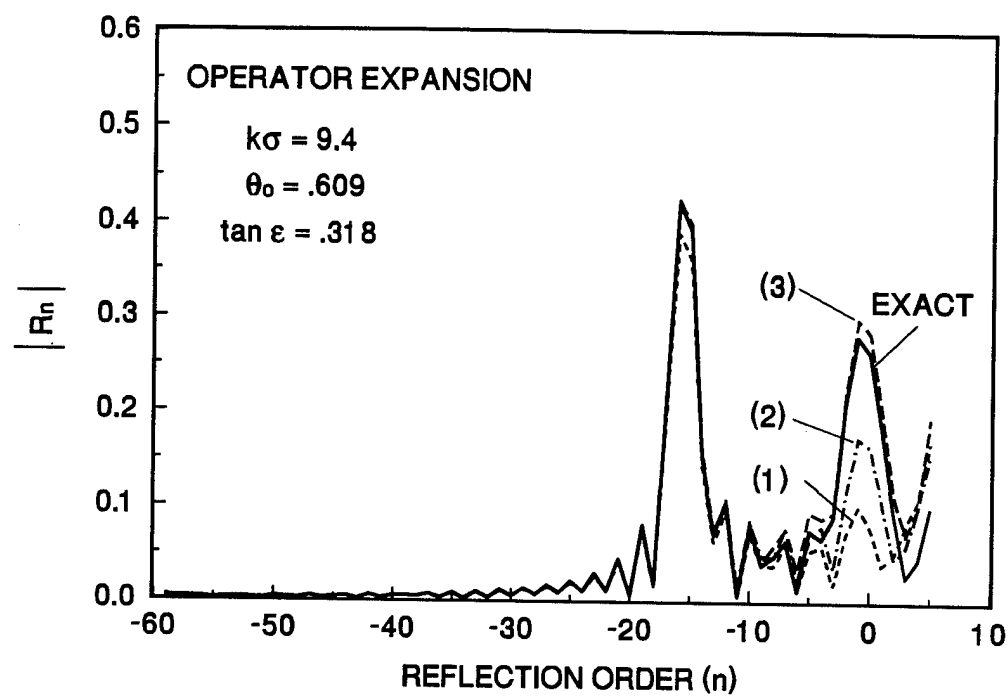
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Figure 5. Magnitude of the reflection coefficient as a function of reflected order predicted by the operator expansion for the case of figure 4.
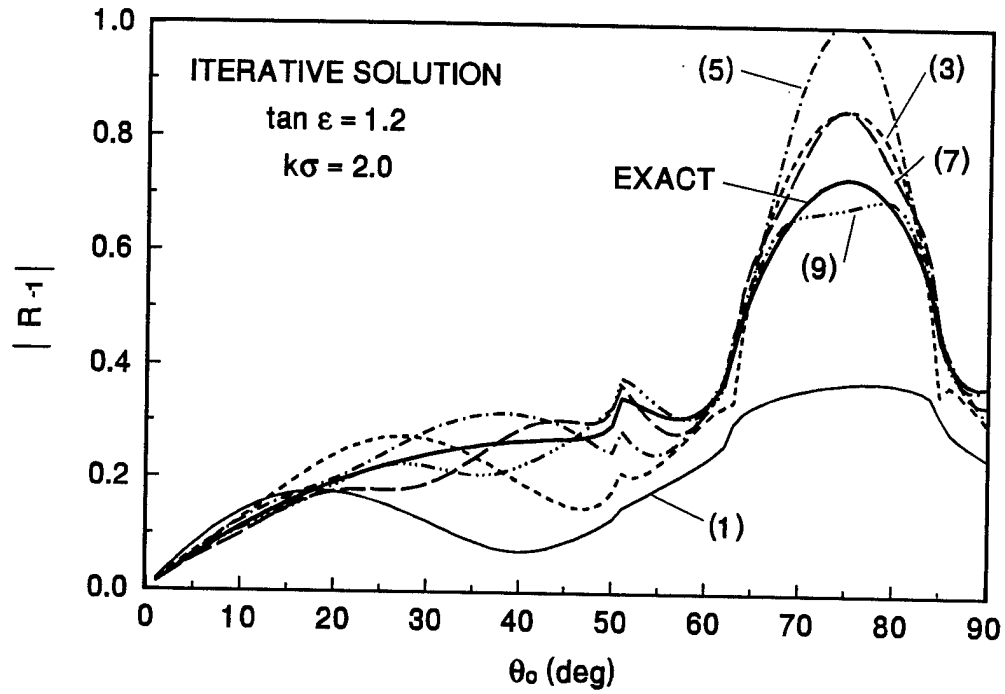
Figure 6. Magnitude of the reflection coefficient R_{-1} as a function of incident grazing angle predicted by the iterative solution for $k\sigma = 2$ and $\tan \epsilon = 1.2$. Again, the numbers in parentheses indicate the order of the slope.
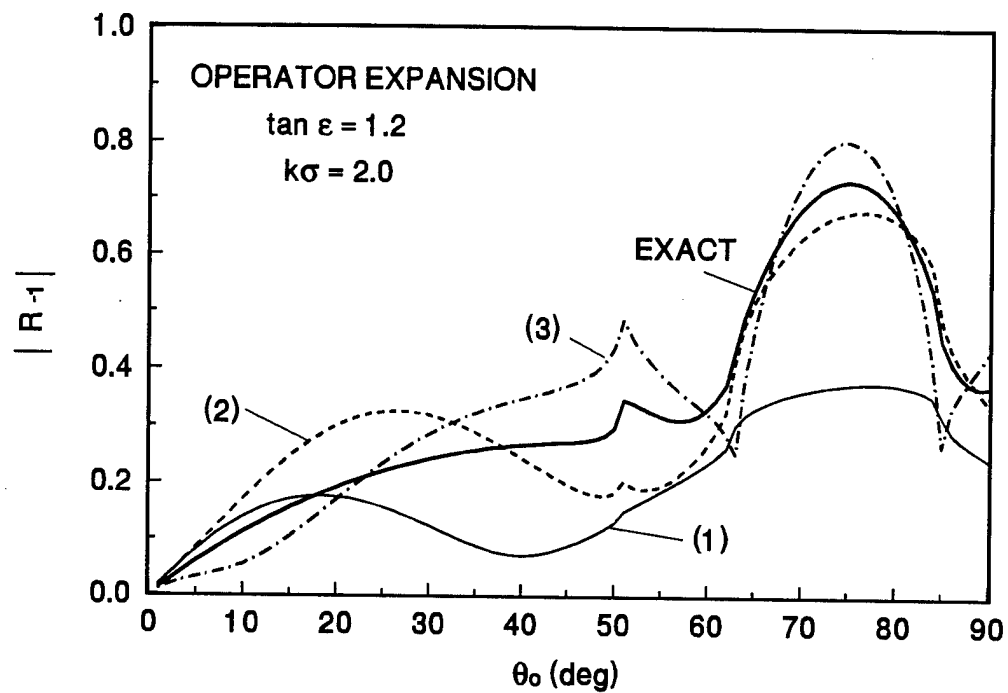
Figure 7. Magnitude of the reflection coefficient R_{-1} as a function of incident grazing angle predicted by the operator expansion for the case of figure 6.

REPORT DOCUMENTATION PAGE

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1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE February 1997	3. REPORT TYPE AND DATES COVERED Technical

4. TITLE AND SUBTITLE

An Iterative Solution of the
Rough-Surface Scattering Problem

5. FUNDING NUMBERS

N00014-95-1-0538

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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217

10. SPONSORING / MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION / AVAILABILITY STATEMENT

Distribution Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

An iterative solution to the problem of scattering from a one-dimensional rough surface is obtained for the Dirichlet boundary condition. The advantages of this method are that bounds for the convergence of the solution may be established and that the solution may be readily iterated to sufficiently high order in the interaction to examine the rate at which it converges. Absolute convergence of the iterative solution is also a sufficient condition for the convergence of the operator expansion method for surfaces on which the slope is everywhere less than unity. A numerical example of scattering from an echelette grating is considered, and bounds for convergence established. It is found that for scattering from such surfaces, the rate at which the iterative solution converges decreases as the surface slope is increased. Corresponding results are found for the operator expansion method.

14. SUBJECT TERMS	15. NUMBER OF PAGES
Scattering from rough interfaces	20
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR